

# Decomposition of Cellular Balleans

*I. V. Protasov, A. Tsvietkova*

**Abstract.** A ballean is a set endowed with some family of its subsets which are called the balls. We postulate the properties of the family of balls in such a way that the balleans can be considered as the asymptotic counterparts of the uniform topological spaces. The isomorphisms in the category of balleans are called asy-morphisms. Every metric space can be considered as a ballean. The ultrametric spaces are prototypes for the cellular balleans. We prove some general theorem about decomposition of a homogeneous cellular ballean in a direct product of a pointed family of sets. Applying this theorem we show that the balleans of two uncountable groups of the same regular cardinality are asy-morphic.

A *ball structure* is a triple  $\mathcal{B} = (X, P, B)$  where  $X, P$  are non-empty sets, and for all  $x \in X$  and  $\alpha \in P$ ,  $B(x, \alpha)$  is a subset of  $X$  which is called a *ball of radius  $\alpha$  around  $x$* . It is supposed that  $x \in B(x, \alpha)$  for all  $x \in X$ ,  $\alpha \in P$ . The set  $X$  is called the *support* of  $\mathcal{B}$ ,  $P$  is called the *set of radii*.

Given any  $x \in X, A \subseteq X, \alpha \in P$ , we put

$$\begin{aligned} B^*(x, \alpha) &= \{y \in X : x \in B(y, \alpha)\}, \\ B(A, \alpha) &= \bigcup_{a \in A} B(a, \alpha), \end{aligned} \tag{1}$$

$$B^*(A, \alpha) = \bigcup_{a \in A} B^*(a, \alpha). \tag{2}$$

A ball structure  $\mathcal{B} = (X, P, B)$  is called a *ballean* (or a *coarse structure*) if

- $\forall \alpha, \beta \in P \exists \alpha', \beta' \in P$  such that  $\forall x \in X$

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \quad B^*(x, \beta) \subseteq B(x, \beta');$$

- $\forall \alpha, \beta \in P \exists \gamma \in P$  such that  $\forall x \in X$

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma);$$

Let  $\mathcal{B}_1 = (X_1, P_1, B_1)$  and  $\mathcal{B}_2 = (X_2, P_2, B_2)$  be balleans.

A mapping  $f : X_1 \rightarrow X_2$  is called a  $\prec$ -mapping if  $\forall \alpha \in P_1 \exists \beta \in P_2$  such that:

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta).$$

A bijection  $f : X_1 \rightarrow X_2$  is called an *asymorphism* between  $\mathcal{B}_1$  and  $\mathcal{B}_2$  if  $f$  and  $f^{-1}$  are  $\prec$ -mappings. In this case  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are called *asymorphic*.

If  $X_1 = X_2$  and the identity mapping  $\text{id} : X_1 \rightarrow X_2$  is an asymorphism, we identify  $\mathcal{B}_1$  and  $\mathcal{B}_2$  and write  $\mathcal{B}_1 = \mathcal{B}_2$ .

For motivation to study balleans, see [1], [2], [3], [4].

Every metric space  $(X, d)$  determines the *metric ballean*  $\mathcal{B}(X, d) = (X, \mathbb{R}^+, B_d)$ , where  $\mathbb{R}^+$  is the set of non-negative real numbers,

$$B_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

A ballean  $\mathcal{B}$  is called *metrizable* if  $\mathcal{B}$  is asymorphic to  $\mathcal{B}(X, d)$  for some metric ballean. By [3, Theorem 2.1], a ballean  $\mathcal{B}$  is metrizable if and only if  $\mathcal{B}$  is connected and the cofinality  $\text{cf}(\mathcal{B}) \leq \aleph_0$ . A ballean  $\mathcal{B} = (X, P, B)$  is *connected* if, for any  $x, y \in X$ , there exists  $\alpha \in P$  such that  $y \in B(x, \alpha)$ . To define  $\text{cf}(\mathcal{B})$ , we use the natural preordering on  $P$ :  $\alpha \leq \beta$  if and only if  $B(x, \alpha) \subseteq B(x, \beta)$  for every  $x \in X$ . A subset  $P'$  is *cofinal* in  $P$  if, for every  $\alpha \in P$ , there exists  $\alpha' \in P'$  such that  $\alpha \leq \alpha'$ , so  $\text{cf}(\mathcal{B})$  is the minimal cardinality of cofinal subsets of  $P$ .

Given an arbitrary ballean  $\mathcal{B} = (X, P, B)$ ,  $x, y \in X$  and  $\alpha \in P$ , we say that  $x, y$  are  $\alpha$ -path connected if there exists a finite sequence  $x_0, x_1, \dots, x_n$ ,  $x_0 = x$ ,  $x_n = y$  such that  $x_{i+1} \in B(x_i, \alpha)$ , for every  $i \in \{0, 1, \dots, n-1\}$ . For any  $x \in X$  and  $\alpha \in P$ , we put

$$B^\square(x, \alpha) = \{y \in X : x, y \text{ are } \alpha\text{-path connected}\}$$

The ballean  $\mathcal{B}^\square = (X, P, B^\square)$  is called the *cellularization* of  $\mathcal{B}$ . A ballean  $\mathcal{B}$  is called *cellular* if  $\mathcal{B}^\square = \mathcal{B}$ . For characterizations of cellular balleans see [3, Chapter 3].

**Example 1.** A metric  $d$  on a set  $X$  is called an *ultrametric* if

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}$$

for all  $x, y, z \in X$ . If  $(X, d)$  is an ultrametric space then the ballean  $\mathcal{B}(X, d)$  is cellular. Moreover, by [3, Theorem 3.1], a ballean  $\mathcal{B}$  is metrizable and cellular if and only if  $\mathcal{B}$  is asymorphic to the metric ballean  $\mathcal{B}(X, d)$  of some ultrametric space  $(X, d)$ .

**Example 2.** Let  $G$  be an infinite group with the identity  $e$ ,  $\kappa$  be an infinite cardinal such that  $\kappa \leq |G|$ ,  $\mathcal{F}(G, \kappa) = \{A \subseteq G : e \in A, |A| < \kappa\}$ .

Given any  $g \in G$  and  $A \in F(G, \kappa)$ , we put  $B(g, A) = gA$  and get the ballean  $\mathcal{B}(G, \kappa) = (G, F(G, \kappa), B)$ . In the case  $\kappa = |G|$ , we write  $\mathcal{B}(G)$  instead of  $\mathcal{B}(G, \kappa)$ . A ballean  $\mathcal{B}(G, \kappa)$  is cellular if and only if either  $\kappa > \aleph_0$  or  $\kappa = \aleph_0$  and  $G$  is locally finite (i.e. every finite subset of  $G$  is contained in some finite subgroup).

**Example 3.** A family of subsets of a group  $G$  is called a *Boolean group ideal* if

- $A, B \in \mathfrak{S} \Rightarrow A \cup B \in \mathfrak{S}$ ;
- $A \in \mathfrak{S}, A' \subset A \Rightarrow A' \in \mathfrak{S}$ ;
- $A, B \in \mathfrak{S} \Rightarrow AB \in \mathfrak{S}, A^{-1} \in \mathfrak{S}$ ;
- $F \in \mathfrak{S}$  for every finite subset  $F$  of  $G$ .

Every Boolean group ideal  $\mathfrak{S}$  on  $G$  determines the ballean  $\mathcal{B}(G, \mathfrak{S}) = (G, \mathfrak{S}, B)$ , where  $B(g, A) = gA$  for all  $g \in G, A \in \mathfrak{S}$ . The ballenans on groups determined by the Boolean group ideals can be considered (see [3, Chapter 6]) as the asymptotic counterparts of the group topologies. A ballean  $\mathcal{B}(G, \mathfrak{S})$  is cellular if and only if  $\mathfrak{S}$  has a base consisting of the subgroups of  $G$ .

A connected ballean  $\mathcal{B} = (X, P, B)$  is called ordinal if there exists a cofinal well-ordered (by  $\leq$ ) subset of  $P$ . Clearly, every metrizable ballean is ordinal.

**Theorem 1.** *Let  $\mathcal{B} = (X, P, B)$  be an ordinal ballean. Then  $\mathcal{B}$  is either metrizable or cellular.*

*Proof.* If  $\text{cf}(\mathcal{B}) \leq \aleph_0$  then  $\mathcal{B}$  is metrizable by theorem 2.1 from [3]. Assume that  $\text{cf}(\mathcal{B}) > \aleph_0$ . Given an arbitrary  $\alpha \in P$ , we choose inductively a sequence  $(\alpha_n)_{n \in \omega}$  in  $P$  such that  $\alpha_0 = \alpha$  and  $B(B(x, \alpha_n), \alpha) \subseteq B(x, \alpha_{n+1})$  for every  $x \in X$ . Since  $\text{cf}(\mathcal{B}) > \aleph_0$ , we can pick  $\beta \in P$  such that  $\beta \geq \alpha_n$  for every  $n \in \omega$ . Then  $B^\square(x, \alpha) \subseteq B(x, \beta)$  for every  $x \in X$ , so  $\mathcal{B}^\square = \mathcal{B}$ .

Let  $\gamma$  be an ordinal,  $\{Z_\lambda : \lambda < \gamma\}$  be a family of non-empty sets. For every  $\lambda < \gamma$  we fix some element  $e_\lambda \in Z_\lambda$  and say that the family  $\{(Z_\lambda, e_\lambda) : \lambda < \gamma\}$  is *pointed*. A *direct product*  $Z = \otimes_{\lambda < \gamma} (Z_\lambda, e_\lambda)$  is the set of all functions  $f : \{\lambda : \lambda < \gamma\} \rightarrow \cup_{\lambda < \gamma} Z_\lambda$  such that  $f(\lambda) \in Z_\lambda$  and  $f(\lambda) = e_\lambda$  for all but finitely many  $\lambda < \gamma$ . We consider the ball structure  $\mathcal{B}(Z) = (Z, \{ \lambda : \lambda < \gamma \}, B)$ , where  $B(f, \lambda) = \{g \in Z : f(\lambda') = g(\lambda') \text{ for all } \lambda' \geq \lambda\}$ . It is easy to verify that  $\mathcal{B}(Z)$  is a cellular ballean.

We say that a ballean  $\mathcal{B}$  is *decomposable in a direct product* if  $\mathcal{B}$  is asyomorphic to  $\mathcal{B}(Z)$  for some direct product  $Z$ .

**Theorem 2.** *Let  $\gamma$  be a limit ordinal,  $\mathcal{B} = (Z, \{\lambda : \lambda < \gamma\}, B)$  be a ballean such that:*

- (i)  $B^\square(x, \alpha) = B(x, \alpha)$  for all  $x \in X, \alpha \in P$ ;
- (ii) if  $\alpha < \beta < \gamma$  then  $B(x, \alpha) \subset B(x, \beta)$  for each  $x \in X$ ;
- (iii) if  $\beta$  is a limit ordinal and  $\beta < \gamma$  then  $B(x, \beta) = \bigcup_{\alpha < \beta} B(x, \alpha)$  for each  $x \in X$ ;
- (iv) there exists a cardinal  $\kappa_0$  such that  $B(x, 0) = \kappa_0$  for each  $x \in X$ ;
- (v) for every  $\alpha < \gamma$  there exists a cardinal  $\kappa_\alpha$  such that every ball of radius  $\alpha + 1$  is a disjoint union of  $\kappa_\alpha$ -many balls of radius  $\alpha$ .

*Then  $\mathcal{B}$  is decomposable in a direct product.*

*Proof.* We fix some set  $Z_0$  of cardinality  $\kappa_0$  and define inductively a family of sets  $\{Z_\alpha, \alpha < \gamma\}$ . If  $\alpha$  is a limit ordinal, we take  $Z_\alpha$  to be a singleton. If  $\alpha = \beta + 1$  we take a set  $Z_\alpha$  of cardinality  $\kappa_\beta$ . For every  $\alpha < \gamma$ , we choose some element  $e_\alpha \in Z_\alpha$ , put  $Z = \bigotimes_{\lambda < \gamma} (Z_\lambda, e_\lambda)$  and show that  $\mathcal{B}$  is asyomorphic to  $\mathcal{B}(Z)$ . To this end we fix some element  $x_0 \in X$  and, for every  $\alpha < \gamma$ , define a mapping  $f_\alpha : B(x_0, \alpha) \rightarrow \bigotimes_{\beta \leq \alpha} (Z_\beta, e_\beta)$  such that, for all  $\beta < \alpha < \gamma$ ,  $f_\alpha|_{B(x_0, \beta)} = f_\beta$  and the inductive limit  $f$  of the family  $\{f_\alpha : \alpha < \gamma\}$  is an asyomorphism between  $\mathcal{B}$  and  $\mathcal{B}(Z)$ . Here we identify  $\bigotimes_{\beta \leq \alpha} (Z_\beta, e_\beta)$  with the corresponding subset of  $\bigotimes_{\beta < \gamma} (Z_\beta, e_\beta)$ .

At the first step we fix some bijection  $f_0 : B(x_0, 0) \rightarrow Z_0$  such that  $f_0(x_0) = e_0$ . Let us assume that, for some  $\alpha < \gamma$ , we have defined the mappings  $\{f_\beta : \beta < \alpha\}$ . If  $\alpha$  is a limit ordinal, we put  $f_\alpha : B(x_0, \alpha) \rightarrow \bigotimes_{\beta < \alpha} (Z_\beta, e_\beta)$  to be an inductive limit of the family  $\{f_\beta : \beta < \alpha\}$ . Since  $Z_\alpha = \{e_\alpha\}$  we can identify  $\bigotimes_{\beta < \alpha} (Z_\beta, e_\beta)$  with  $\bigotimes_{\beta \leq \alpha} (Z_\beta, e_\beta)$ , so  $f_\alpha : B(x_0, \alpha) \rightarrow \bigotimes_{\beta \leq \alpha} (Z_\beta, e_\beta)$ . If  $\alpha = \beta + 1$ , by cellularity of  $\mathcal{B}$ , there exists a subset  $Y \subseteq B(x_0, \alpha), x_0 \in Y$  such that  $B(x_0, \alpha)$  is a disjoint union of the family  $\{B(y, \beta) : y \in Y\}$ . For every  $y \in Y$ , we can repeat the inductive procedure of construction of  $f_\alpha : B(x_0, \beta) \rightarrow \bigotimes_{\lambda \leq \beta} (Z_\lambda, e_\lambda)$  to define a mapping  $f'_{\beta, y} : B(y, \beta) \rightarrow \bigotimes_{\lambda \leq \beta} (Z_\lambda, e_\lambda)$ . Thus we fix some bijection  $h : Y \rightarrow Z_\alpha, h(x_0) = e_\alpha$  and put  $f_{\beta, y}(x) = (f'_{\beta, y}(x), h(y)), x \in B(y, \beta)$ . At last, given any  $x \in B(x_0, \alpha)$ , we choose  $y \in Y$  such that  $x \in B(y, \beta)$  and put  $f_\alpha(x) = f_{\beta, y}(x)$ . By the construction of  $f$  as an inductive limit of the family  $\{f_\alpha : \alpha < \gamma\}$ , given any  $x \in X$  and  $\alpha < \gamma$ , we have  $f(B(x, \alpha)) = B(f(x), \alpha)$  so  $f$  is an asyomorphism.

In the next two corollaries and Theorem 3  $\mathcal{B}(G)$  is a ballean defined in Example 2.

**Corollary 1.** *Let  $G$  be a countable locally finite group. Then  $\mathcal{B}(G)$  is decomposable in a direct product of finite sets.*

*Proof.* We write  $G$  as a union  $G = \cup_{n < \omega} G_n$  of an increasing chain of finite groups. Clearly,  $\mathcal{B}(G)$  is isomorphic to the ballean  $\mathcal{B} = (G, \omega, B)$  where  $B(g, n) = gG_n$ . We put  $\kappa_0 = |G_0|$ ,  $\kappa_{n+1} = |G_{n+1} : G_n|$  and apply Theorem 2.

**Corollary 2.** *Let  $G$  be an uncountable group of regular cardinality  $\gamma$ . Then  $\mathcal{B}(G)$  is decomposable in a direct product.*

*Proof.* We write  $G$  as a union  $G = \cup_{\alpha < \gamma} G_\alpha$  of an increasing chain of subgroups such that  $|G_0| = \aleph_0$ ,  $|G_\alpha| < \gamma$  and  $G_\alpha = \cup_{\beta < \alpha} G_\beta$  for every limit ordinal  $\alpha$ . Since  $\gamma$  is regular, every subset  $F \subset G$ ,  $|F| < |G|$  is contained in some subgroup  $G_\alpha$ . It follows that  $\mathcal{B}(G)$  is isomorphic to the ballean  $\mathcal{B} = (G, \gamma, B)$ , where  $B(g, \alpha) = gG_\alpha$ . Apply Theorem 2.

**Theorem 3.** *Let  $G, H$  be two uncountable groups of the same regular cardinality  $\gamma$ . Then  $\mathcal{B}(G)$  and  $\mathcal{B}(H)$  are isomorphic.*

*Proof.* We consider two cases.

*Case 1:*  $\gamma$  is a limit cardinal. We choose an increasing family  $\{G_\alpha : \alpha < \gamma\}$  of subgroups of  $G$  such that  $G = \cup_{\alpha < \gamma} G_\alpha$ ,  $|G_0| = \aleph_0$ ,  $|G_{\alpha+1}| = |G_\alpha|^+$  and  $G_\beta = \cup_{\alpha < \beta} G_\alpha$  for every limit ordinal  $\beta$ . Put  $\kappa_\alpha = |G_\alpha|^+$ ,  $\alpha < \gamma$ . By Theorem 2,  $\mathcal{B}(G)$  is isomorphic to  $\mathcal{B}(Z)$  where the direct product  $Z$  is defined by the family of cardinals  $\{\kappa_\alpha : \alpha < \gamma\}$ . Since  $H$  admits a filtration  $H = \cup_{\alpha < \gamma} H_\alpha$  with the same family  $\{\kappa_\alpha : \alpha < \gamma\}$  of parameters,  $\mathcal{B}(H)$  is also isomorphic to  $\mathcal{B}(Z)$ .

*Case 2:*  $\gamma = \lambda^+$  for some cardinal  $\lambda$ . We write  $G$  as a union  $G = \cup_{\alpha < \gamma} G_\alpha$  of an increasing family of subgroups such that  $|G_\alpha| = \lambda$ ,  $|G_{\alpha+1} : G_\alpha| = \lambda$  for every  $\alpha < \gamma$ , and  $G_\beta = \cup_{\alpha < \beta} G_\alpha$  for every limit ordinal  $\beta$ . Put  $\kappa_\alpha = \lambda$  for every  $\alpha < \gamma$ . By Theorem 2,  $\mathcal{B}(G)$  is isomorphic to  $\mathcal{B}(Z)$ , where  $Z$  is defined by the family of parameters  $\{\kappa_\alpha : \alpha < \gamma\}$ . Since  $H$  admits a filtration with the same family of parameters,  $\mathcal{B}(H)$  is also isomorphic to  $\mathcal{B}(Z)$ .

This completes the proof.

It should be mentioned that Theorem 3 does not hold for countable groups. By [2, Theorem 10.6], there exists a family  $\mathcal{F}$  of countable locally finite groups such that any two groups from  $\mathcal{F}$  are non-isomorphic and  $|\mathcal{F}| = 2^{\aleph_0}$ .

We do not know if Corollary 2 and Theorem 3 are true for groups of singular cardinalities.

## References

- [1] A. N. Dranishnikov, *Asymptotic topology*, Russian Math Surveys 55 (2000), no. 6, 1085-1129.
- [2] Protasov I., Banakh T. *Ball Structures and Colorings of Graphs and Groups*, Mathematical Studies Monograph Series, 11. L'viv: VNTL Publishers, 2003.
- [3] Ihor Protasov and Michael Zarichnyi, *General Asymptology*, Mathematical Studies Monograph Series, 12. L'viv: VNTL Publishers, 2007.
- [4] John Roe, *Lectures on Coarse Geometry*, University Lecture Series, 31. Providence, RI: AMS, 2003.

I.V. Protasov (*protasov@unicyb.kiev.ua*)  
Department of Cybernetics, Kyiv National University, Volodimirska 64,  
Kiev 01033, UKRAINE

Anastasiia Tsvietkova (*tsvietkova@math.utk.edu*)  
Department of Mathematics, University of Tennessee, 121 Ayres Hall,  
Knoxville, Tennessee 37996-1300, USA